

Black hole evaporation in a spherically symmetric non-commutative space-time

Elisabetta Di Grezia[†], Giampiero Esposito[‡], Gennaro Miele[‡]

[†]Università Statale di Bergamo, Facoltà di Ingegneria, Viale Marconi 5, 24044 Dalmine (Bergamo); INFN, Sezione di Milano, Via Celoria 16, 20133 Milano

[‡]INFN, Sezione di Napoli, and Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio 6, 80126 Napoli, Italy

Abstract. Recent work in the literature has studied the quantum-mechanical decay of a Schwarzschild-like black hole, formed by gravitational collapse, into almost-flat space-time and weak radiation at a very late time. The relevant quantum amplitudes have been evaluated for bosonic and fermionic fields, showing that no information is lost in collapse to a black hole. On the other hand, recent developments in noncommutative geometry have shown that, in general relativity, the effects of noncommutativity can be taken into account by keeping the standard form of the Einstein tensor on the left-hand side of the field equations and introducing a modified energy-momentum tensor as a source on the right-hand side. Relying on the recently obtained noncommutativity effect on a static, spherically symmetric metric, we have considered from a new perspective the quantum amplitudes in black hole evaporation. The general relativity analysis of spin-2 amplitudes has been shown to be modified by a multiplicative factor F depending on a constant non-commutativity parameter and on the upper limit R of the radial coordinate. Limiting forms of F have been derived which are compatible with the adiabatic approximation.

1. Introduction

Theoretical research in black hole physics has witnessed, over the last few years, an impressive amount of new ideas and results on at least four main areas:

- (i) The problem of information loss in black holes, after the suggestion in Ref. [1] that quantum gravity is unitary and information is preserved in black hole formation and evaporation.
- (ii) The related series of papers in Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10], concerned with evaluating quantum amplitudes for transitions from initial to final states, in agreement with a picture where information is not lost, and the end state of black hole evaporation is a combination of outgoing radiation states.
- (iii) The approach in Refs. [11, 12, 13], according to which black holes create instead a vacuum matter charge to protect themselves from the quantum evaporation.

(iv) The work in Ref. [14] where the authors, relying upon the previous findings in Ref. [15], consider a non-commutative radiating Schwarzschild black hole, and find that non-commutativity cures usual problems encountered in trying to describe the latest stage of black hole evaporation.

We have been therefore led to study how non-commutativity would affect the analysis of quantum amplitudes in black hole evaporation performed in Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10]. Following Ref. [14], we assume that non-commutativity of space-time can be encoded in the commutator of operators corresponding to space-time coordinates, i.e. (the integer D below is even)

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad \mu, \nu = 1, 2, \dots, D, \quad (1)$$

where the antisymmetric matrix $\theta^{\mu\nu}$ is taken to have block-diagonal form

$$\theta^{\mu\nu} = \text{diag}(\theta_1, \dots, \theta_{D/2}), \quad (2)$$

with

$$\theta_i = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \forall i = 1, 2, \dots, D/2, \quad (3)$$

the parameter θ having dimension of length squared and being constant. As shown in Ref. [15], the constancy of θ leads to a consistent treatment of Lorentz invariance and unitarity. The authors of Ref. [14] solve the Einstein equations with mass density of a static, spherically symmetric, smeared particle-like gravitational source as (hereafter we work in $G = c = \hbar = 1$ units)

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}} e^{-\frac{r^2}{4\theta}}, \quad (4)$$

which therefore plays the role of matter source. Their resulting spherically symmetric metric is

$$ds^2 = - \left[1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \right] dt^2 + \left[1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \right]^{-1} dr^2 \\ + r^2(d\Theta^2 + \sin^2\Theta d\phi^2), \quad (5)$$

where we use the lower incomplete gamma function [14]

$$\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \equiv \int_0^{\frac{r^2}{4\theta}} \sqrt{t} e^{-t} dt. \quad (6)$$

In this picture, one deals with a mass distribution

$$m(r) \equiv \frac{2M}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right), \quad (7)$$

while M is the total mass of the source [14]. This mass function satisfies the equation

$$m'(r) = 4\pi r^2 \rho_\theta(r),$$

formally analogous to the general relativity case [10].

The work in Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10] studies instead the quantum-mechanical decay of a Schwarzschild-like black hole, formed by gravitational collapse, into almost-flat space-time and weak radiation at a very late time. The spin-2 gravitational perturbations split into parts with odd and even parity, and one can isolate suitable variables which can be taken as boundary data on a final spacelike hypersurface Σ_F . The main idea is then to consider a complexified classical boundary-value problem where T is rotated into the complex: $T \rightarrow |T|e^{-i\alpha}$, for $\alpha \in]0, \pi/2]$, and evaluate the corresponding classical Lorentzian action $S_{\text{class}}^{(2)}$ to quadratic order in metric perturbations. The genuinely Lorentzian quantum amplitude is recovered by taking the limit as $\alpha \rightarrow 0^+$ of the semiclassical amplitude $e^{iS_{\text{class}}^{(2)}}$ [2, 6, 10].

Section II studies the differential equations obeyed by radial modes within the framework of the adiabatic approximation, and Sec. III obtains the resulting orthogonality relation in the presence of a non-vanishing non-commutativity parameter θ . Section IV derives the effect of θ on the expansion of the pure-gravity action functional, which can be used in the evaluation of quantum amplitudes along the lines of Refs. [2]-[10], while concluding remarks are presented in Sec. V.

2. Equations for radial modes

The analysis in Ref. [10] holds for any spherically symmetric Lorentzian background metric

$$ds^2 = -e^{b(r,t)} dt^2 + e^{a(r,t)} dr^2 + r^2(d\Theta^2 + \sin^2 \Theta d\phi^2), \quad (8)$$

the even modes $\xi_{2lm}^{(+)}(r, t)$ and odd modes $\xi_{2lm}^{(-)}(r, t)$ being built from a Fourier-type decomposition, i.e. [10]

$$\xi_{2lm}^{(+)}(r, t) = \int_{-\infty}^{\infty} dk a_{2klm}^{(+)} \xi_{2kl}^{(+)}(r) \frac{\sin kt}{\sin kT}, \quad (9)$$

and

$$\xi_{2lm}^{(-)}(r, t) = \int_{-\infty}^{\infty} dk a_{2klm}^{(-)} \xi_{2kl}^{(-)}(r) \frac{\cos kt}{\sin kT}, \quad (10)$$

where the radial functions $\xi_{2kl}^{(\pm)}$ obey the following second-order differential equation:

$$e^{-a} \frac{d}{dr} \left(e^{-a} \frac{d\xi_{2kl}^{(\pm)}}{dr} \right) + \left(k^2 - V_l^{\pm}(r) \right) \xi_{2kl}^{(\pm)} = 0, \quad (11)$$

where, on defining $\lambda \equiv \frac{(l+2)(l-1)}{2}$, the potential terms are given by [10]

$$V_l^{+}(r) = e^{-a(r,t)} \frac{2[\lambda^2(\lambda+1)r^3 + 3\lambda^2 m r^2 + 9]m^2 r + 9m^3}{r^3(\lambda r + 3m)^2}, \quad (12)$$

and

$$V_l^{-}(r) = e^{-a(r,t)} \left(\frac{l(l+1)}{r^2} - \frac{6m}{r^3} \right), \quad (13)$$

respectively. In the expansion of the gravitational action to quadratic order, it is of crucial importance to evaluate the integral

$$I(k, k', l, R) \equiv \int_0^R e^{a(r,t)} \xi_{2kl}^{(+)}(r) \xi_{2k'l}^{(+)}(r) dr, \quad (14)$$

since [10]

$$S_{\text{class}}^{(2)}[(h_{ij}^{(\pm)})_{lm}] = \frac{\pm 1}{32\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^l \frac{(l-2)!}{(l+2)!} \int_0^R e^a \xi_{2lm}^{(\pm)} \left(\frac{\partial}{\partial t} \xi_{2lm}^{(\pm)*} \right)_{t=T} dr. \quad (15)$$

For this purpose, we bear in mind the limiting behaviours [10]

$$\xi_{2kl}^{(\pm)} \sim \text{const} \times (kr)^{l+1} + O((kr)^{l+3}) \text{ as } r \rightarrow 0, \quad (16)$$

$$\xi_{2kl}^{(\pm)}(r) \sim z_{2kl}^{(\pm)} e^{ikr_s} + z_{2kl}^{(\pm)*} e^{-ikr_s} \text{ as } r \rightarrow \infty, \quad (17)$$

where Eq. (16) results from imposing regularity at the origin, r_s is the Regge–Wheeler tortoise coordinate [10, 16]

$$r_s(r) \equiv r + 2M \log(r - 2M), \quad (18)$$

while $z_{2kl}^{(\pm)}$ are complex constants. Indeed, it should be stressed that non-commutativity can smear plane waves into Gaussian wave packets. Thus, in a fully self-consistent analysis, the Fourier modes in Eq. (9), (10), and their asymptotic form in Eq. (16), (17), should be modified accordingly. However, this task goes beyond the aims of the present paper, and is deferred to a future publication.

With this understanding, we can now exploit Eq. (11) to write the equations (hereafter, we write for simplicity of notation ξ_{kl} rather than $\xi_{2kl}^{(\pm)}$, and similarly for V_l rather than V_l^{\pm})

$$e^a \xi_{k'l} \left[e^{-a} \frac{d}{dr} \left(e^{-a} \frac{d}{dr} \xi_{kl} \right) + (k^2 - V_l) \xi_{kl} \right] = 0, \quad (19)$$

$$e^a \xi_{kl} \left[e^{-a} \frac{d}{dr} \left(e^{-a} \frac{d}{dr} \xi_{k'l} \right) + (k'^2 - V_l) \xi_{k'l} \right] = 0. \quad (20)$$

According to a standard procedure, if we subtract Eq. (20) from Eq. (19), and integrate the resulting equation from $r = 0$ to $r = R$, we obtain

$$(k^2 - k'^2) \int_0^R e^a \xi_{kl} \xi_{k'l} dr = \int_0^R \left[\xi_{kl} \frac{d}{dr} \left(e^{-a} \frac{d}{dr} \xi_{k'l} \right) - \xi_{k'l} \frac{d}{dr} \left(e^{-a} \frac{d}{dr} \xi_{kl} \right) \right] dr. \quad (21)$$

The desired integral (14) is therefore obtained from Eq. (21), whose right-hand side is then completely determined from the limiting behaviours in Eqs. (16) and (17), i.e.

$$\int_0^R e^a \xi_{kl} \xi_{k'l} dr = \left\{ \frac{1}{(k^2 - k'^2)} \left[\xi_{kl} e^{-a} \left(\frac{d}{dr} \xi_{k'l} \right) - \xi_{k'l} e^{-a} \left(\frac{d}{dr} \xi_{kl} \right) \right]_{r=0}^{r=R} \right\}, \quad (22)$$

where, on going from Eq. (21) to Eq. (22), we have exploited the vanishing coefficient that weights the integral

$$\int_0^R e^{-a} \left(\frac{d}{dr} \xi_{kl} \right) \left(\frac{d}{dr} \xi_{k'l} \right) dr,$$

resulting from two contributions of equal magnitude and opposite sign. By virtue of Eq. (16), $r = 0$ gives vanishing contribution to the right-hand side of Eq. (22), while the contribution of first derivatives of radial functions involves also

$$\left. \frac{dr_s}{dr} \right|_{r=R} = \frac{R}{(R - 2M)}. \quad (23)$$

3. Generalized orthogonality relation

Note now that our metric (5) is a particular case of the spherically symmetric metric (8), since our a and b functions are independent of time. More precisely, unlike the full Vaidya space-time, where in the region containing outgoing radiation the mass function varies extremely slowly with respect both to t and to r [9], we consider a “hybrid” scheme where the mass function depends on r only for any fixed value of the non-commutativity parameter θ . We can therefore write

$$e^{-a} = 1 - \frac{4M}{r\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \quad (24)$$

in our non-commutative spherically symmetric model, where the function in curly brackets in Eq. (22) reads as

$$\begin{aligned} & \left(1 - \frac{4M}{R\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{R^2}{4\theta}\right)\right) \frac{R}{(R - 2M)} \frac{1}{(k^2 - k'^2)} \times i \left[(k' - k) z_{kl} z_{k'l} e^{i(k+k')r_s(R)} \right. \\ & \left. + (k - k') z_{kl}^* z_{k'l}^* e^{-i(k+k')r_s(R)} - (k + k') z_{kl} z_{k'l}^* e^{i(k-k')r_s(R)} + (k + k') z_{kl}^* z_{k'l} e^{i(k'-k)r_s(R)} \right]. \end{aligned}$$

At this stage, we exploit one of the familiar limits that can be used to express the Dirac δ , i.e. [10]

$$\lim_{r_s \rightarrow \infty} \frac{e^{i(k \pm k')r_s}}{(k \pm k')} = i\pi \delta(k \pm k'), \quad (25)$$

to find

$$\int_0^R e^a \xi_{kl} \xi_{k'l} dr = 2\pi |z_{kl}|^2 F(R, \theta) \left(\delta(k + k') + \delta(k - k') \right), \quad (26)$$

having defined

$$F(R, \theta) \equiv \frac{R}{(R - 2M)} \left[1 - \frac{4M}{R\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{R^2}{4\theta}\right) \right]. \quad (27)$$

4. Effect of θ and expansion of the action functional

Since θ has dimension length squared as we said after Eq. (3), we can define the non-commutativity-induced length scale

$$L \equiv 2\sqrt{\theta}. \quad (28)$$

Moreover, we know that our results only hold in the adiabatic approximation, i.e. when both m' and \dot{m} are very small. The latter condition is obviously satisfied because our

mass function in Eq. (7) is independent of time. The former amounts to requiring that (hereafter we set $w \equiv R/L$, while $R_s \equiv 2M$)

$$m'(R) = \frac{2}{\sqrt{\pi}} \frac{R_s}{L} e^{-w^2} w^2 \ll 1. \quad (29)$$

The condition (29) is satisfied provided that either

(i) $w \rightarrow \infty$ or $w \rightarrow 0$, i.e. $R \gg L$ or $R \ll L$;

(ii) or at $R = L$ such that

$$m'(R = L) = m'(w = 1) = \frac{2}{\sqrt{\pi}} \frac{R_s}{L} e^{-1} \ll 1, \quad (30)$$

and hence for $\frac{R_s}{L} \ll \frac{e\sqrt{\pi}}{2}$.

Furthermore, at finite values of the non-commutativity parameter θ , our $w \equiv \frac{R}{L}$ is always much larger than 1 in Eq. (27) if R is very large, and hence we can exploit the asymptotic expansion of the lower incomplete γ -function in this limit [17, 18], i.e.

$$\gamma\left(\frac{3}{2}, w^2\right) = \Gamma\left(\frac{3}{2}\right) - \Gamma\left(\frac{3}{2}, w^2\right) \sim \frac{1}{2}\sqrt{\pi} \left[1 - e^{-w^2} \sum_{p=0}^{\infty} \frac{w^{1-2p}}{\Gamma\left(\frac{3}{2} - p\right)}\right]. \quad (31)$$

By virtue of Eqs. (27) and (31), we find

$$F(R, \theta) \equiv F(R, L) \sim 1 + \frac{R_s}{(R - R_s)} e^{-w^2} \sum_{p=0}^{\infty} \frac{w^{1-2p}}{\Gamma\left(\frac{3}{2} - p\right)}. \quad (32)$$

Equation (32) describes the asymptotic expansion of the correction factor F when $R \gg L$.

In the opposite regime, i.e. for θ so large that $(R/L) \ll 1$ despite that R tends to ∞ , one has [18]

$$F(R, L) \sim \frac{R}{(R - R_s)} \left[1 - \frac{4}{3\sqrt{\pi}} \frac{R_s}{R} w^3 \left(1 - \frac{7}{5} w^2\right)\right]. \quad (33)$$

Last, but not least, if R and L are comparable, the lower-incomplete γ -function in Eq. (27) cannot be expanded, and we find, bearing in mind that $R_s/L \ll 1$ from Eq. (30), the limiting form

$$F(R, L) \sim 1 + \frac{R_s}{L} \left(1 - \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, 1\right)\right) + O((R_s/L)^2). \quad (34)$$

We therefore conclude that a θ -dependent correction to the general relativity analysis in Ref. [10] does indeed arise from non-commutative geometry. In particular, the expansion of the action to quadratic order in perturbative modes takes the form (cf. Ref. [10])

$$\begin{aligned} S_{\text{class}}^{(2)} &= \frac{F(R, L)}{16} \sum_{l=2}^{\infty} \sum_{m=-l}^l \sum_{P=\pm 1} \frac{(l-2)!}{(l+2)!} \times \\ &\times \int_0^{\infty} dk \, k |z_{2klP}|^2 |a_{2klmP} + P a_{2,-klmP}|^2 \cot kT, \end{aligned} \quad (35)$$

where the function $F(R, L)$ (see Eq. (27)) takes the limiting forms (32) and (33), respectively, depending on whether $w \gg 1$ or $w \ll 1$, while $P = \pm 1$ for even (respectively odd) metric perturbations. Our “correction” $F(R, L)$ to the general relativity analysis is non-vanishing provided that one works at very large but finite values of R . In the limit as $R \rightarrow \infty$, one has instead

$$\lim_{R \rightarrow \infty} F(R, L) = 1, \quad (36)$$

which means that, at infinite distance from the Lorentzian singularity of Schwarzschild geometry, one cannot detect the effect of a non-commutativity parameter.

5. Concluding remarks

Our paper has investigated the effect of non-commutative geometry on the recent theoretical analysis of quantum amplitudes in black hole evaporation, following the work in Refs. [1], [2]-[10] (for other developments, see for example the recent work in Refs. [19, 20, 21, 22]). For this purpose, we have considered an approximate scheme where the background space-time is static and spherically symmetric, with mass function depending on the radial coordinate only for any fixed value of the non-commutativity parameter θ .

Within this framework, we find [23] that the general relativity analysis of spin-2 amplitudes is modified by a multiplicative factor F defined in Eq. (27). Its limiting forms for $R \gg L$ or $R \ll L$ or $R \cong L$ are given by Eqs. (32), (33) and (34), respectively. Within this framework, *unitarity is preserved, and the end state of black hole evaporation is a combination of outgoing radiation states* (see section 1).

An outstanding open problem is whether one can derive a time-dependent spherically symmetric background metric which incorporates the effects of non-commutative geometry. This would make it possible to improve the present comparison with the results in Refs. [2]-[10], where the Vaidya space-time was taken as the background geometry. More recently, for the scalar wave equation in a non-commutative spherically symmetric space-time, we have built the associated conformal infinity [24], and the analysis of the wave equation has been reduced to the task of solving an inhomogeneous Euler–Poisson–Darboux equation. The scalar field has been found to have an asymptotic behaviour with a fall-off going on rather more slowly than in flat space-time, in full qualitative agreement with general relativity [25].

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References

- [1] Hawking S W 2005 *Phys. Rev. D* **72** 084013

- [2] Farley A N St J and D'Eath P D 2004 *Phys. Lett. B* **601** 184
- [3] Farley A N St J and D'Eath P D 2005 *Phys. Lett. B* **613** 181
- [4] Farley A N St J and D'Eath P D 2005 *Class. Quantum Grav.* **22** 2765
- [5] Farley A N St J and D'Eath P D 2005 *Class. Quantum Grav.* **22** 3001
- [6] Farley A N St J and D'Eath P D 2005 'Quantum amplitudes in black-hole evaporation. I. Complex approach' (gr-qc/0510028)
- [7] Farley A N St J and D'Eath P D 2005 'Quantum amplitudes in black-hole evaporation. II. Spin-0 amplitude' (gr-qc/0510029)
- [8] Farley A N St J and D'Eath P D 2006 *Phys. Lett. B* **634** 419
- [9] Farley A N St J and D'Eath P D 2006 *Gen. Rel. Grav.* **38** 425
- [10] Farley A N St J and D'Eath P D 2006 *Ann. Phys. (N.Y.)* **321** 1334
- [11] Vilkovisky G A 2006 *Ann. Phys. (N.Y.)* **321** 2717
- [12] Vilkovisky G A 2006 *Phys. Lett. B* **634** 456
- [13] Vilkovisky G A 2006 *Phys. Lett. B* **638** 523
- [14] Nicolini P, Smailagic A, and Spallucci E 2006 *Phys. Lett. B* **632** 547
- [15] Smailagic A and Spallucci E 2004 *J. Phys. A* **37** 7169
- [16] Regge T and Wheeler J A 1957 *Phys. Rev.* **108** 1063
- [17] Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* (New York: Dover)
- [18] Spallucci E, Smailagic A and Nicolini P 2006 *Phys. Rev. D* **73** 084004
- [19] Kar S and Majumdar S 2006 *Int. J. Mod. Phys. A* **21** 6087
- [20] Kar S and Majumdar S 2006 *Phys. Rev. D* **74** 066003
- [21] Kar S 2006 *Phys. Rev. D* **74** 126002
- [22] Alberghi G L, Casadio R, Galli D, Gregori D, Tronconi A, and Vignoni V 2006 'Probing quantum gravity effects in black holes at LHC' (hep-ph/0601243).
- [23] Di Grezia E, Esposito G, and Miele G 2006 *Class. Quantum Grav.* **23** 6425
- [24] Di Grezia E, Esposito G, and Miele G 2007 arXiv:0705.0242 [hep-th]
- [25] Schmidt B G and Stewart J M 1979 *Proc. Roy. Soc. Lond. A* **367** 503